Graph Neural Networks for Causal Inference Under Network Confounding

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Marc [illegible]

Abstract. [illegible]

1 Introduction

U VA

U VA

x

x **m** **y**

$\boldsymbol{X}$

$\boldsymbol{A}$ A

1.2 Related Literature

i n o ′ K 1 i o
A i $n(i, 1$

2 Setup

$\mathcal{N}_n$ $1, \ldots, n$ $\boldsymbol{A}$
i $\mathcal{N}_n$ ′ $(\varepsilon_i, \nu_i$ $\mathbb{R}^{d_\varepsilon}$ $\mathbb{R}^{d_\nu}$ ′ X_i $\mathbb{R}^{d_x}$
′

$$Y_i \quad g_n(i, \boldsymbol{D}, \boldsymbol{X}, \boldsymbol{A}, \boldsymbol{\varepsilon} \qquad D_i \quad h_n(i, \boldsymbol{X}, \boldsymbol{A}, \boldsymbol{\nu} \;,$$

′ $\boldsymbol{X}$ $(X_i$ $_{i=1}^{n}$ i ′ X_i' $\boldsymbol{Y}$ $\boldsymbol{D}$ $\boldsymbol{\varepsilon}$
$\boldsymbol{\nu}$ ′ ′ $(g_n, h_n$ $_{n \in \mathbb{N}}$
$g_n($ $\mathbb{R}$ $h_n($ $0, 1$
$(\boldsymbol{Y}, \boldsymbol{D}, \boldsymbol{X}, \boldsymbol{A}$ ′ $(\boldsymbol{A}, \boldsymbol{X}, \boldsymbol{\varepsilon}, \boldsymbol{\nu}$
$(\boldsymbol{X}, \boldsymbol{A}$ ′

′ ″
$(\boldsymbol{A}, \boldsymbol{X}, \boldsymbol{\varepsilon}, \boldsymbol{\nu}$ ′ ″
$h_n($ ″ $g_n($
$g_n($
$h_n($ ″ ″ Y_i D_i
′ ′
′

E p 1 ′

$$Y_i \quad \alpha + \beta \frac{\sum_{j=1}^{n} A_{ij} Y_j}{\sum_{j=1}^{n} A_{ij}} + \frac{\sum_{j=1}^{n} A_{ij} Z_j'}{\sum_{j=1}^{n} A_{ij}} \gamma + Z_j' \delta + \varepsilon_i,$$

Z_i $(D_i, X_i'$ $'$ β

1 n

A

$$Y \quad \frac{\alpha}{1 \quad \beta} 1 + Z\delta + \gamma\beta \sum_{k=0} \beta^{k} \tilde{A}^{k+1} Z + \sum_{k=0} \beta^{k} \tilde{A}^{k} \varepsilon.$$

Y_i $g_n(i, D, X, A, \varepsilon$

E p

$$D_i \quad 1 \quad \alpha + \beta \frac{\sum_{j=1}^{n} A_{ij} D_j}{\sum_{j=1}^{n} A_{ij}} + \frac{\sum_{j=1}^{n} A_{ij} Z_j'}{\sum_{j=1}^{n} A_{ij}} \gamma + Z_i'\delta + \nu_i \quad 0 \quad .$$

U

$(X, A, \nu$

D D_i $h_n(i, X, A, \nu$

i $(\nu_i, X, A$

D_j $\sigma_j(X, A$

D_j 1 D_i $h_n(i, X, A, \nu_i$

E p

D_i i Y_i

$$Y_i \quad g_n(D_{\mathcal{N}(i,K)}, \varepsilon_i \qquad D_i \quad 1 \; W_i'\gamma \quad \nu_i \; ,$$

W_i $(X, A$ K

$$Y_i(d \quad g_n(i, d, X, A, \varepsilon \; .$$

$Y_i(d$ D_i

$(\boldsymbol{X}, \boldsymbol{A}$

ε $\boldsymbol{\nu}$

Assumption 1 U o n $\mathbf{n}$ $\mathbb{N}$, ε $\boldsymbol{\nu}$ $\boldsymbol{X}, \boldsymbol{A}$

A $(\boldsymbol{X}, \boldsymbol{A}$

U VA $\boldsymbol{\varepsilon}_i$

$\boldsymbol{\nu}_i$ $\mathbf{X}_i$

$\mathbf{S}$ $\mathcal{N}_{\mathbf{n}}$ $\boldsymbol{D}_{\mathbf{S}}$ $(\mathbf{D}_i$ $_{i \in \mathbf{S}}$ $\boldsymbol{X}_{\mathbf{S}}$

$\boldsymbol{A}_{\mathbf{S}}$

s

s $y_n(s$

s Y_i

i i

i

D_i

E p

$\sup_n y_n(s$ $C|\beta|^s$

C 0

$\sup_n \eta_n(s$ s

Y_i

D $D_{\mathcal{N}(i,K)}$ $y_n(s$ $c\,1$ s K

c W_i X_i

$\eta_n(s$ 0 s

2.1 Related Literature

U VA

$$Y_i \quad g(D_i, X_i, \varepsilon_i \qquad \varepsilon_i \quad D_i \quad X_i.$$

$$T_i \quad f_n(i, D, A \qquad W_i \quad q_n(i, X, A$$

$f_n($ $q_n($ n

o u n A

$Y_i = g(T_i, W_i, \varepsilon_i)$ ε_i T_i W_i,

T_i W_i

T_i W_i

$$\tilde{T}_i = \left(D_i, \sum_{j=1}^{n} A_{ij} D_j\right) \qquad \tilde{W}_i = \left(X_i, \sum_{j=1}^{n} A_{ij}, \frac{\sum_{j=1}^{n} A_{ij} X_j}{\sum_{j=1}^{n} A_{ij}}\right).$$

T_i Y_i D

V

$D_{\mathcal{N}(i,1)}$

W_i

K

$D_{\mathcal{N}(i,K)}$ K A

A

T_i

W_i U

W_i X_i

X A

$(X, A$

$\mathbf{t} \quad (1, 0 \qquad \mathbf{t}' \quad (0, 0$

$\mathcal{M}_{\mathbf{n}}$

$\mathbf{n}(\mathbf{i}, 1 \qquad |\mathcal{N}(\mathbf{i}, 1\,|$

$\mathbb{R}_{+} \quad 0$

$\mathcal{M}_{\mathbf{n}} \quad \mathbf{i}$

$(\quad +$

$(X, A, \nu$ $\mathbf{T_i}$ $\mathbf{Y_i}$

D

$\tau(\mathbf{t}, \mathbf{t}'$

U $\tau(\mathbf{t}, \mathbf{t}'$

$\tau(\mathbf{t}, \mathbf{t}'$

ν_i

$\mathbf{i}$ $\mathbf{D_i}$ $\mathbf{h_n}(\mathbf{i}, X, A, \nu_\mathbf{i}$

$(X, A$

V

$\tau(\mathbf{t}, \mathbf{t}'$

A A A

$$\hat{\tau}_i(t, t') = \frac{\mathbb{1}\{T_i = t\}(Y_i - \hat{\mu}_t(i, X, A))}{\hat{p}_t(i, X, A)} + \hat{\mu}_t(i, X, A) - \frac{\mathbb{1}\{T_i = t'\}(Y_i - \hat{\mu}_{t'}(i, X, A))}{\hat{p}_{t'}(i, X, A)} - \hat{\mu}_{t'}(i, X, A).$$

3.1 Architecture

$\mathbf{X}_i$

$\Gamma($ $\boldsymbol{\mu}($, $\boldsymbol{\sigma}($, $\Sigma($, $\min($, $\max($

$\Gamma($ $\Gamma_1($

$\Gamma_1($ $\boldsymbol{\mu}($ $\boldsymbol{\sigma}($ $\Sigma($ $\min($ $\max($

$\mathbf{n}(\mathbf{i}, 1$

$\Gamma_1($

$| \; |$

$$\mathbf{S}(\, , \boldsymbol{\alpha} \quad \left(\frac{\log(| \; | \; +1}{\boldsymbol{\delta}} \right)^{\boldsymbol{\alpha}}, \quad \boldsymbol{\delta} \quad \frac{1}{\mathbf{n}} \sum_{\mathbf{i}=1} \log \left(\sum_{\mathbf{j}=1} \mathbf{A}_{\mathbf{ij}} \; +1 \right), \quad \boldsymbol{\alpha} \quad 1, 1 \, .$$

$\boldsymbol{\alpha}$ $\boldsymbol{\alpha}$ 1 $\boldsymbol{\alpha}$

1 $\boldsymbol{\alpha}$ 0

□(

$$\hat{\mathbf{f}}_{\mathrm{GNN}} \quad \operatorname*{argmin}_{\mathbf{f}\in\mathcal{F}_{\mathrm{GNN}}(\mathbf{L})} \; {}_{\mathbf{i}\in} \; \ddot{\mathbf{y}}$$

A ʾ **f** $\mathcal{F}_{\mathrm{GNN}}($**L** i i

i $(\pi_i(X \; , \pi_i(A$

ropo on 1 $u \; o \; o \; n \; \mathbf{n} \; \mathbb{N} \; n \; u \; on \; \pi,$

$$\mathsf{f}_n(\mathsf{i}, D, A \quad \mathsf{f}_n(\pi(\mathsf{i} \; , \pi(D \; , \pi(A \; ,$$

$$\mathsf{g}_n(\mathsf{i}, D, X, A, \varepsilon \quad \mathsf{g}_n(\pi(\mathsf{i} \; , \pi(D \; , \pi(X \; , \pi(A \; , \pi(\varepsilon \; , \quad n$$

$$\phi_{t,t'}(i \quad \frac{1 \; T_i \quad t \; (Y_i \quad \mu_t(i, X, A}{p_t(i, X, A} + \mu_t(i, X, A$$
$$\frac{1 \; T_i \quad t' \; (Y_i \quad \mu_{t'}(i, X, A}{p_{t'}(i, X, A} \quad \mu_{t'}(i, X, A \quad \tau(t, t' ,$$

$i \quad \mathcal{M}_n$

$$\sigma_n^2 \quad \mathrm{V} \quad \frac{1}{m_n} \sum_{i \in \mathcal{M}_n} \phi_{t,t'}(i \quad X, A \; .$$

A u p on M n p 4 u o n
n $\mathbb{N}$, i $\mathcal{N}_n$, n d $0, 1$ n, $|Y_i(d|^p$ X, A M
$\underline{\pi}, \pi$ $(0, 1$ u $\hat{p}_t(i, X, A, p_t(i, X, A$ $\underline{\pi}, \pi$ n

$\phi_{t,t'}(i \quad {}_{i=1}^{n} \quad \psi$

$\phi_{t,t^1}($i $_{i=1}^{n}$ ψ

$\Lambda_n($s 2M M

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A

A

or n $\phi_{t,t'}(i$ n $\tau(t, t'$ n n on o $\phi_{t,t'}(i$ $\tau_i(t, t'$

$|Y_i$ T_i $t, X,$

$\hat{\sigma}^2$ ' '' ' '

' A

5 Approximate Sparsity

A ' **L**

$(\boldsymbol{X}_{\mathcal{N}(\mathbf{i},\mathbf{L})}$

$(W_i)_{i=1}^n$ ν $(\nu_i)_{i=1}^n$

$$V_i(W,\nu;\theta) = \alpha + \beta \frac{\sum_{j=1}^n A_{ij} W_j}{\sum_{j=1}^n A_{ij}} + \delta \frac{\sum_{j=1}^n A_{ij} X_j}{\sum_{j=1}^n A_{ij}} + \gamma X_j + \nu_i + \frac{\sum_{j=1}^n A_{ij} \nu_j}{\sum_{j=1}^n A_{ij}}$$

θ $(\alpha, \beta, \delta, \gamma)$ $(Y_i)_{i=1}^n$

Y_i $V_i(Y, \varepsilon; \theta_y)$ θ_y $(0.5, 0.8, 10, 1)$ $(D_i)_{i=1}^n$

D_i 1 $V_i(D, \nu; \theta_d)$ 0 θ_d $(0.5, 1.5, 1, 1)$

$(D_i^0)_{i=1}^n$ D_i^0 1 $V_i(0, \nu; \theta_d)$ 0

$\nu_i + \sum_{j=1}^n A_{ij} \nu_j$ $\sum_{j=1}^n A_{ij}$

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A

n 2 Y_i T_i t, X, A

T_i t n $1000, 2000, 4000$

6.2 Nonparametric Estimators

A $\Gamma_2($

L $1,$

L 1

	$L=1$			$L=2$			$L=3$		
n / H	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
$\hat{\tau}(1,0)$	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
C[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible] C[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
W $\hat{\tau}(1,0)$	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
W C[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
W	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible] C[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

si u ations, he esti and is $\boldsymbol{\tau}(1,0) = 0$, treated $\approx$

A **L** 2

L $2, 4, 8, \dots$

L 2

L

7.1 Comparison with He and Song (2024)

$\tau(t, t'$

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A

n 4413

[0.01, 0.99

A

$T_i^{(1)}$

	ADM	GNN			GLM		
		Layer	Layer	Layer	Order	Order	Order
Leader case							
G_{ee}	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
G_{sc}	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
G_{all}	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
Leader adopter case							
G_{ee}	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
G_{sc}	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
G_{all}	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
Adopter case							
G_{ee}	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
G_{sc}	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
G_{all}	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

$n = 4413$

$\mathbf{T_i}^{(1)}$

L

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A

$\mathbf{T_i}^{(2)}$ $2+$

$\mathbf{T_i}^{(1)}$

A

A

$\mathbf{T_i}^{(1)}$

A Additional Results on GNNs

A

$$\frac{1}{m_n}\sum_{i\in\mathcal{M}_n}\left(\hat{p}_t(i,X,A) - p_t(i,X,A)\right)^2 = o_p(n^{-1/2}).$$

U

$$\frac{1}{m_n}\sum_{i\in\mathcal{M}_n}\left(\hat{p}_t(i,X,A) - p_t(i,X_{\mathcal{N}(i,L)},A_{\mathcal{N}(i,L)})\right)^2 = o_p(n^{-1/2}).$$ A

$\hat{p}_t(i,X,A)$ L $(X_{\mathcal{N}(i,L)}, A_{\mathcal{N}(i,L)})$ $p_t(i,X_{\mathcal{N}(i,L)},A_{\mathcal{N}(i,L)})$

A

A

$$\frac{1}{n}\sum_{i=1}^{n}\left(\hat{p}_t(i,X,A) - p_t(i,X_{\mathcal{N}(i,L)},A_{\mathcal{N}(i,L)})\right)^2 \leq C\left(\frac{WL\log R}{n}\log n + \frac{\log\log n + \gamma}{n} + [illegible]^2\right)$$ A

$1 - e^{-\gamma}$ W C

n R

ψ

A A

W, R, L, n A

A.1 WL Function Class

n n

$(\boldsymbol{X}, \boldsymbol{A}$, $(\boldsymbol{X}', \boldsymbol{A}'$ **f** $\mathbf{f}(\boldsymbol{X}, \boldsymbol{A}$ $\neq$ $\mathbf{f}(\boldsymbol{X}', \boldsymbol{A}'$

W n W

$(\boldsymbol{X}, \boldsymbol{A}$

$\mathbf{H}^2$

$(\mathbf{h}, \mathbf{h}'$ $\boldsymbol{\rho}(\mathcal{F}$ $\mathbf{f}(\mathbf{h}$ $\mathbf{f}(\mathbf{h}'$ $\mathbf{f}$ $\mathcal{F}.$

$\mathcal{E}, \mathcal{F}$ $\mathbf{H}$ $\mathcal{E}$ o

n $\mathcal{F}$ $\boldsymbol{\rho}(\mathcal{F}$ $\boldsymbol{\rho}(\mathcal{E}$

A $\mathcal{E}$

$\mathcal{F}$

$\mathcal{F}$ $\mathbf{H}$ $\mathcal{E}$

$\mathbf{f}$ $_{\mathrm{L},\mathbf{L}}$ $(\boldsymbol{X}, \boldsymbol{A}$ $\Sigma^{\mathbf{n}}$

$\mathbf{L}$ $\mathcal{C}(\mathbf{H}$

$\mathbf{H}$ $\mathbf{L}$ $\mathbb{N}$ W un on

$$\mathcal{F}\ _{\mathrm{L}}(\mathbf{L} \quad \mathbf{f} \quad \mathcal{C}(\mathbf{H} : \boldsymbol{\rho}(\mathbf{f}\ _{\mathrm{L},\mathbf{L}} \quad \boldsymbol{\rho}(\mathbf{f} \quad .$$

$(\boldsymbol{X}, \boldsymbol{A}$

$\mathbf{L}$

$\mathbf{p}_{\mathbf{t}}($ $\boldsymbol{\mu}_{\mathbf{t}}($

$\mathcal{F}$ L(**L** **L**

$\mathcal{F}_{\text{GNN}}$ (**L**

A

A

T L

A

f $\mathcal{F}_{\text{GNN}}$ (L

$$(X, A \quad \underbrace{f(i, X, A), \ldots, f(i, X, A)}_{n \text{ times}} \quad \mathbb{R}^n$$

L +1

$\varphi_{L+1}(h_i^{(L)}, h_j^{(L)} : j \quad \mathcal{N}_n$

$\mathcal{F}_{\text{GNN}}$ (L A

X_i $\rho(\mathcal{F}_{\text{GNN}}$ (L $\rho($ f $_{L,L}$ L

L ■

A.2 Disadvantages of Depth

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L L

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n 1 n 1

B Verifying §8 Assumptions

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A

max $y_n(s$ 2 , $\psi_n(s$

($c(1$ 4 p ^{-1}s c 0 p A A

$$\sup_{n} \max_{i \in \mathcal{N}_n} |\mathcal{N}_{\boldsymbol{A}}(i, s| \quad Cs^{d}$$

C 0 $d \succ 1$

$D'_{\mathbf{B}}$ $(\mathbf{D}'_{\mathbf{j}}$ $_{\mathbf{j}\in\mathbf{B}}$ $\mathbf{B}$ $\mathcal{N}_{\mathbf{n}}$ U

$$\mathbf{p_t}(\mathbf{i}, X, A) \quad (\mathbf{D}'_{\mathbf{i}} + (\mathbf{D_i} \quad \mathbf{D}'_{\mathbf{i}} \quad |\mathbf{a}, \mathbf{b}, \quad \mathbf{V}'_{\mathbf{i}} + (\mathbf{V_i} \quad \mathbf{V}'_{\mathbf{i}} \quad |\alpha, \beta \quad X, A$$

$$(\mathbf{D}'_{\mathbf{i}} \quad |\mathbf{a} \quad , \mathbf{b} + \quad , \quad \mathbf{V}'_{\mathbf{i}} \quad |\alpha \quad , \beta + \quad X, A$$

$$+ \mathbf{I}\ (|\mathbf{D_i} \quad \mathbf{D}'_{\mathbf{i}}| \quad X, A \quad \mathbf{m} \quad + \quad (|\mathbf{V_i} \quad \mathbf{V}'_{\mathbf{i}}| \quad X, A \quad \mathbf{n} \cdot$$

$\mathbf{R}_0$

$(\mathbf{D}'_{\mathbf{i}} \quad |\mathbf{a}, \mathbf{b}$

$$|\mathbf{p_t}(\mathbf{i}, X, A \quad \mathbf{p_t}(\mathbf{i}, X_{\mathcal{N}(\mathbf{i}, \mathbf{r}_\lambda(\mathbf{s}+1))}, A_{\mathcal{N}(\mathbf{i}, \mathbf{r}_\lambda(\mathbf{s}+1))}) | \quad \boldsymbol{\lambda}_\mathbf{n}(\mathbf{s} \ +1 \quad +2\mathbf{R}_0.$$

roo o

$$\boldsymbol{\mu}_\mathbf{t}(\mathbf{i}, X, A \quad |\mathbf{Y_i} 1_\mathbf{i}(\mathbf{t} \quad X, A \quad \mathbf{p_t}(\mathbf{i}, X, A$$

$$\mathbf{B} \quad \mathcal{N}(\mathbf{i}, \mathbf{s} \quad \mathbf{Y}_\mathbf{i}' \quad \mathbf{g}_{\mathbf{n}(\mathbf{i},\mathbf{s})}(\mathbf{i}, D_\mathbf{B}', X_\mathbf{B}, A_\mathbf{B}, \varepsilon_\mathbf{B}$$

$$| \quad |\mathbf{Y_i} 1_\mathbf{i}(\mathbf{t} \quad X, A \quad |\mathbf{Y}_\mathbf{i}' 1_\mathbf{i}(\mathbf{t} \quad X, A | \quad \boldsymbol{\gamma}_\mathbf{n}(\mathbf{s} \quad +\Lambda_\mathbf{n}(\mathbf{i}, \mathbf{s} \ \mathbf{n}(\mathbf{i}, \mathbf{s} \ \boldsymbol{\eta}_\mathbf{n}(\mathbf{s}$$

A ▼

$|\mathbf{R}_1|$ $\boldsymbol{\lambda}_\mathbf{n}$

$|\mathbf{R}_1| \quad \mathbf{y_n}(\mathbf{s} \quad + \Lambda_{\mathbf{n}}(\mathbf{i}, \mathbf{s} \; \mathbf{n}(\mathbf{i}, \mathbf{s} \; \eta_{\mathbf{n}}(\mathbf{s}$ ■

C n $\mathbf{Y}_{\mathbf{i}}', \mathbf{D}_{\mathbf{i}}'$ n n $\mathbf{1}_{\mathbf{i}}(\mathbf{t}\,' \quad 1 \; \mathbf{D}_{\mathbf{i}}' \quad \mathbf{d}, \sum_{\mathbf{j}=1}^{\mathbf{n}} \mathbf{A}_{\mathbf{ij}} \mathbf{D}_{\mathbf{j}}'$
n A on , , , n , **C** 0 o n
$\mathbf{n} \quad \mathbb{N}, \mathbf{i} \quad \mathcal{N}_{\mathbf{n}}, \quad n \quad \mathbf{s} \succ 0,$

$$|\mathbf{Y}_{\mathbf{i}}| \mathbf{1}_{\mathbf{i}}(\mathbf{t} \quad \mathbf{1}_{\mathbf{i}}(\mathbf{t}\,'| \quad \boldsymbol{X}, \boldsymbol{A} \quad \mathbf{C}\,(1 \; + \mathbf{n}(\mathbf{i}, 1 \quad \eta_{\mathbf{n}}(\mathbf{s} \; .$$

Proof. $\mathbf{a}, \mathbf{b}, \boldsymbol{\alpha}, \boldsymbol{\beta},$ $\mathbf{V}_{\mathbf{i}} \quad \sum_{\mathbf{j}=1}^{\mathbf{n}} \mathbf{A}_{\mathbf{ij}} \mathbf{D}_{\mathbf{j}}$
$\mathbf{V}_{\mathbf{i}}' \quad \sum_{\mathbf{j}=1}^{\mathbf{n}} \mathbf{A}_{\mathbf{ij}} \mathbf{D}_{\mathbf{j}}'$ $\mathcal{C}$ $|\mathbf{D}_{\mathbf{i}} \quad \mathbf{D}_{\mathbf{i}}'|$, $|\mathbf{V}_{\mathbf{i}} \quad \mathbf{V}_{\mathbf{i}}'|$

$$|\mathbf{Y}_{\mathbf{i}}| \mathbf{1}_{\mathbf{i}}(\mathbf{t} \quad \mathbf{1}_{\mathbf{i}}(\mathbf{t}\,'| \quad \boldsymbol{X} \quad \boldsymbol{x}, \boldsymbol{A} \quad \boldsymbol{a}$$
$$|\mathbf{Y}_{\mathbf{i}}| \mathbf{1}_{\mathbf{i}}(\mathbf{t} \quad \mathbf{1}_{\mathbf{i}}(\mathbf{t}\,'| \quad \mathcal{C}, \boldsymbol{X} \quad \boldsymbol{x}, \boldsymbol{A} \quad \boldsymbol{a} \; + \mathbf{C} \quad (\mathcal{C}^{\mathbf{c}} \quad \boldsymbol{X} \quad \boldsymbol{x}, \boldsymbol{A} \quad \boldsymbol{a}$$

C 0 A A

$$\mathbf{1}_{\mathbf{i}}(\mathbf{t} \quad 1 \quad \mathbf{D}_{\mathbf{i}} \quad |\mathbf{a}, \mathbf{b} \; , \; \mathbf{V}_{\mathbf{i}} \quad |\boldsymbol{\alpha}, \boldsymbol{\beta} \qquad \mathbf{1}_{\mathbf{i}}(\mathbf{t}\,' \quad 1 \quad \mathbf{D}_{\mathbf{i}}' \quad |\mathbf{a}, \mathbf{b} \; , \; \mathbf{V}_{\mathbf{i}}' \quad |\boldsymbol{\alpha}, \boldsymbol{\beta} \quad .$$

U $\mathcal{C}$

$$1 \; \mathbf{D}_{\mathbf{i}} \quad |\mathbf{a}, \mathbf{b} \; , \; \mathbf{V}_{\mathbf{i}} \quad |\boldsymbol{\alpha}, \boldsymbol{\beta} \; (\quad 1 \; \mathbf{D}_{\mathbf{i}}' \; + (\mathbf{D}_{\mathbf{i}} \quad \mathbf{D}_{\mathbf{i}}' \quad |\mathbf{a}, \mathbf{b} \; , \; \mathbf{V}_{\mathbf{i}}' \; + (\mathbf{V}_{\mathbf{i}} \quad \mathbf{V}_{\mathbf{i}}' \quad |\boldsymbol{\alpha}, \boldsymbol{\beta}$$
$$1 \; \mathbf{D}_{\mathbf{i}}' \quad |\mathbf{a} \quad , \mathbf{b} \; + \; , \; \mathbf{V}_{\mathbf{i}}' \quad |\boldsymbol{\alpha} \quad \mathbf{s}$$

$\mathbb{R}^{\mathbf{d}}$ $(\mathbf{f}$ $\mathbf{f}$ $\mathcal{L}_{\mathbf{d}}$

$$\mathcal{P}_{\mathbf{n}}(\mathbf{h}, \mathbf{h}'; \mathbf{s} \quad (\mathbf{H}, \mathbf{H}' : \mathbf{H}, \mathbf{H}' \quad \mathcal{N}_{\mathbf{n}}, \mathbf{H} \quad \mathbf{h}, \mathbf{H}' \quad \mathbf{h}', \square_{A}(\mathbf{H}, \mathbf{H}' \succ \mathbf{s} \ .$$

D n on C 1 A $\mathbf{Z}_{\mathbf{i}}$ $_{\mathbf{i}=1}^{\mathbf{n}}$ *on on* $\boldsymbol{\psi}$ *n n n* $\mathcal{F}_{\mathbf{n}}$ $\mathbf{C}$ $(0,$ $\mathcal{F}_{\mathbf{n}}$ $\boldsymbol{\psi}_{\mathbf{n}}(\mathbf{s}$ $_{\mathbf{s},\mathbf{n}\in\mathbb{N}}$ $\boldsymbol{\psi}_{\mathbf{n}}(0$ 1 $\mathbf{n}$

$$| \quad (\mathbf{f}(Z_{\mathbf{H}}, \mathbf{f}'(Z_{\mathbf{H}'} \quad | \quad \mathbf{C}\mathbf{h}\mathbf{h}'(\|\mathbf{f}\|_{\infty} + \quad (\mathbf{f} \quad (\|\mathbf{f}'\|_{\infty} + \quad (\mathbf{f}' \quad \boldsymbol{\psi}_{\mathbf{n}}(\mathbf{s}$$

$\mathbf{n}, \mathbf{h}, \mathbf{h}'$ $\mathbb{N}$ $\mathbf{s}$ 0 $\mathbf{f}$ $\mathcal{L}_{\mathbf{h}}$ $\mathbf{f}'$ $\mathcal{L}_{\mathbf{h}'}$ $(\mathbf{H}, \mathbf{H}'$ $\mathcal{P}_{\mathbf{n}}(\mathbf{h}, \mathbf{h}'; \mathbf{s}$ $\boldsymbol{\psi}_{\mathbf{n}}(\mathbf{s}$ *n n o n* $\mathbf{Z}_{\mathbf{i}}$ $_{\mathbf{i}=1}^{\mathbf{n}}$

C *n* A *u on , , , n , n n , o ny* $\mathbf{t}, \mathbf{t}'$ $\mathcal{T},$ $\boldsymbol{\phi}_{\mathbf{t},\mathbf{t}'}(\mathbf{i}$ $_{\mathbf{i}=1}^{\mathbf{n}}$ *on on* $\boldsymbol{\psi}$ *n n n* $(X, A$ *n on n n o n* $\boldsymbol{\psi}_{\mathbf{n}}(\mathbf{s}$ *n n*

Proof. $\mathcal{F}_{\mathbf{n}}$ $\boldsymbol{\sigma}$ $(X, A$ $(\mathbf{h}, \mathbf{h}'$ $\mathbb{N}$ $\mathbb{N}$ $(\mathbf{f}, \mathbf{f}'$ $\mathcal{L}_{\mathbf{h}}$ $\mathcal{L}_{\mathbf{h}'}$ $\mathbf{s}$ 0 $(\mathbf{H}, \mathbf{H}'$ $\mathcal{P}_{\mathbf{n}}(\mathbf{h}, \mathbf{h}'; \mathbf{s}$ $\mathbf{Z}_{\mathbf{i}}$ $\boldsymbol{\phi}_{\mathbf{t},\mathbf{t}'}(\mathbf{i}$ $Z_{\mathbf{H}}$ $(\mathbf{Z}_{\mathbf{i}}$ $_{\mathbf{i}\in\mathbf{H}}$ $\boldsymbol{\xi}$ $\mathbf{f}(Z_{\mathbf{H}}$ $\boldsymbol{\zeta}$ $\mathbf{f}'(Z_{\mathbf{H}'}$

$$\mathbf{D}_{\mathbf{i}}^{(\mathbf{s})} \quad \mathbf{h}_{\mathbf{n}(\mathbf{i},\mathbf{s})}(\mathbf{i}, X_{\mathcal{N}(\mathbf{i},\mathbf{s})}, A_{\mathcal{N}(\mathbf{i},\mathbf{s})}, \nu_{\mathcal{N}(\mathbf{i},\mathbf{s})} \ .$$

$D_{\mathcal{N}(\mathbf{i},\mathbf{s}')}^{(\mathbf{s})}$ $(\mathbf{D}_{\mathbf{j}}^{(\mathbf{s})}$ $_{\mathbf{j}\in\mathcal{N}(\mathbf{i},\mathbf{s}')}$

$$\mathbf{1}_{\mathbf{i}}^{(\mathbf{s})}(\mathbf{t} \quad 1 \ \mathbf{f}_{\mathbf{n}(\mathbf{i},\mathbf{s}/2)}(\mathbf{i}, D_{\mathcal{N}(\mathbf{i},\mathbf{s}/2)}^{(\mathbf{s}/2)}, A_{\mathcal{N}(\mathbf{i},\mathbf{s}/2)} \quad \mathbf{t} \ ,$$
$$\mathbf{Y}_{\mathbf{i}}^{(\mathbf{s})} \quad \mathbf{g}_{\mathbf{n}(\mathbf{i},\mathbf{s}/2)}(\mathbf{i}, D_{\mathcal{N}(\mathbf{i},\mathbf{s}/2)}^{(\mathbf{s}/2)}, X_{\mathcal{N}(\mathbf{i},\mathbf{s}/2)}, A_{\mathcal{N}(\mathbf{i},\mathbf{s}/2)}, \varepsilon_{\mathcal{N}(\mathbf{i},\mathbf{s}/2)} \ ,$$
$$\mathbf{Z}_{\mathbf{i}}^{(\mathbf{s})} \quad 1_{\mathbf{i}}^{(\mathbf{s})}(\mathbf{t} \ (\mathbf{Y}_{\mathbf{i}}^{(\mathbf{s})} \quad \boldsymbol{\mu}_{\mathbf{t}}(\mathbf{i}, X, A$$

A $(\mathbf{Z}_{\mathbf{i}}^{(\mathbf{s}/2,\boldsymbol{\xi})} \quad {}_{\mathbf{i}\in\mathbf{H}} \quad (\mathbf{Z}_{\mathbf{j}}^{(\mathbf{s}/2,\boldsymbol{\zeta})} \quad {}_{\mathbf{j}\in\mathbf{H}^1} \quad \mathcal{F}_{\mathbf{n}}$

$$
\begin{aligned}
| \quad (\boldsymbol{\xi},\boldsymbol{\zeta} \quad \mathcal{F}_{\mathbf{n}} \,| \quad & | \quad (\boldsymbol{\xi} \quad \boldsymbol{\xi}^{(\mathbf{s}/2)},\boldsymbol{\zeta} \quad \mathcal{F}_{\mathbf{n}} \,| \;+| \quad (\boldsymbol{\xi}^{(\mathbf{s}/2)},\boldsymbol{\zeta} \quad \boldsymbol{\zeta}^{(\mathbf{s}/2)} \quad \mathcal{F}_{\mathbf{n}} \,| \\
& 2\|\mathbf{f}'\|_\infty \quad |\,|\boldsymbol{\xi} \quad \boldsymbol{\xi}^{(\mathbf{s}/2)}| \quad \mathcal{F}_{\mathbf{n}} \quad +2\|\mathbf{f}\|_\infty \quad |\,|\boldsymbol{\zeta} \quad \boldsymbol{\zeta}^{(\mathbf{s}/2)}| \quad \mathcal{F}_{\mathbf{n}} \\
& 2 \;\mathbf{h}\|\mathbf{f}'\|_\infty \quad (\mathbf{f} \quad +\mathbf{h}'\|\mathbf{f}\|_\infty \quad (\mathbf{f}' \quad \max_{\mathbf{i}\in\mathcal{N}}
\end{aligned}
$$

$\mathcal{L}_{\mathbf{h}} \quad \mathcal{L}_{\mathbf{h}^1} \;\; \mathbf{s} \quad 0 \;\; (\mathbf{H}, \mathbf{H}' \quad \mathcal{P}_{\mathbf{n}}(\mathbf{h}, \mathbf{h}'; \mathbf{s}$

$$\mathbf{Y}_{\mathbf{i}}^{(\mathbf{s})} \quad \mathbf{g}_{\mathbf{n}(\mathbf{i},\mathbf{s})}(\mathbf{i}, \boldsymbol{D}_{\mathcal{N}(\mathbf{i},\mathbf{s})}, \boldsymbol{X}_{\mathcal{N}(\mathbf{i},\mathbf{s})}, \boldsymbol{A}_{\mathcal{N}(\mathbf{i},\mathbf{s})}, \varepsilon_{\mathcal{N}(\mathbf{i},\mathbf{s})} \;,$$

$\boldsymbol{\xi} \quad \mathbf{f}((\mathbf{Y}_{\mathbf{i}} \;{}_{\mathbf{i}\in\mathbf{H}} \quad \boldsymbol{\zeta} \quad \mathbf{f}'((\mathbf{Y}_{\mathbf{i}} \;{}_{\mathbf{i}\in\mathbf{H}^1} \quad \boldsymbol{\xi}^{(\mathbf{s})} \quad \mathbf{f}((\mathbf{Y}_{\mathbf{i}}^{(\mathbf{s})} \;{}_{\mathbf{i}\in\mathbf{H}} \quad \boldsymbol{\zeta}^{(\mathbf{s})} \quad \mathbf{f}'((\mathbf{Y}_{\mathbf{i}}^{(\mathbf{s})} \;{}_{\mathbf{i}\in\mathbf{H}^1}$
A

$$\begin{aligned} | \quad (\boldsymbol{\xi}, \boldsymbol{\zeta} \;\; \mathcal{F}'_{\mathbf{n}} | \quad & | \quad (\boldsymbol{\xi} \quad \boldsymbol{\xi}^{(\mathbf{s}/2)}, \boldsymbol{\zeta} \;\; \mathcal{F}'_{\mathbf{n}} | + | \quad (\boldsymbol{\xi}^{(\mathbf{s}/2)}, \boldsymbol{\zeta} \quad \boldsymbol{\zeta}^{(\mathbf{s}/2)} \;\; \mathcal{F}'_{\mathbf{n}} | \\ & 2\|\mathbf{f}'\|_\infty \;\; |\,|\boldsymbol{\xi} \quad \boldsymbol{\xi}^{(\mathbf{s}/2)}| \;\; \mathcal{F}'_{\mathbf{n}} \;\; + 2\|\mathbf{f}\|_\infty \;\; |\,|\boldsymbol{\zeta} \quad \boldsymbol{\zeta}^{(\mathbf{s}/2)}| \;\; \mathcal{F}'_{\mathbf{n}} \\ & 2 \;\mathbf{h}\|\mathbf{f}'\|_\infty \quad (\mathbf{f} \quad + \mathbf{h}'\|\mathbf{f}\|_\infty \quad (\mathbf{f}' \;\; \max_{\mathbf{i}\in\mathcal{N}_n} \;\; |\,|\mathbf{Y}_{\mathbf{i}} \quad \mathbf{Y}_{\mathbf{i}}^{(\mathbf{s}/2)}| \;\; \mathcal{F}'_{\mathbf{n}} \\ & 2 \;\mathbf{h}\|\mathbf{f}'\|_\infty \quad (\mathbf{f} \quad + \mathbf{h}'\|\mathbf{f}\|_\infty \quad (\mathbf{f}' \;\; \boldsymbol{\gamma}_{\mathbf{n}}(\mathbf{s} \;\; 2 \;, \end{aligned}$$

A $\boldsymbol{\psi}$

A

$\hat{p}_t(i, X, A$ $\quad$ C, C' 0 R_{1t}^2

$$\frac{1}{m_n} \sum_{i \in \mathcal{M}_n} \sum_{j \in \mathcal{M}_n} (Y_i \quad \mu_i \; (Y_j \quad \mu_j \quad D, X, A \quad 1_i(t \; 1_j(t \; ($$

$\hat{\mathsf{p}}_{\mathsf{t}}(\mathsf{i}, X, A$

$\Delta_{\mathsf{i}}(\mathsf{t}$ $(\hat{\mu}_{\mathsf{t}}(\mathsf{i}$ $\mu_{\mathsf{t}}(\mathsf{i}$ $\mathsf{p}_{\mathsf{t}}(\mathsf{i}$ $1_{\mathsf{i}}(\mathsf{t}$

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GNNs for Network Confounding

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